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# A NOTE ON MAXIMAL ORDERS OVER KRULL DOMAINS

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## 0. Introduction

The classical Mori–Nagata theorem (stating that the integral closure of a Noetherian domain is a Krull domain) is recently generalized to rings satisfying a polynomial identity in the following result by M. Chamarie:

**Theorem 0.1** [1]. *If $\Lambda$ is a Noetherian prime p.i.-ring with center $R$ and ring of quotients $\Sigma$, then there exists an intermediate ring $\Lambda \subset \Lambda' \subset \Sigma$ which is a maximal order with center $R^{\sim}$ (the complete integral closure of $R$) which is a Krull domain.*

Unlike in the commutative case, this 'integral closure' is by no means unique. This difficulty prompts the following question:

**Question A.** If $\Lambda$ is a maximal order over a Krull domain $R$, with ring of quotients $\Sigma$ (which is a central simple algebra over $K$, the field of fractions of $R$), is it possible to describe all other maximal $R$-orders in $\Sigma$ by means of 'invariants' of $\Lambda$?

In this paper we provide a positive answer to this question using cohomology of the sheaf of normalizing elements of $\Lambda$ (introduced in [3]). Furthermore, we will apply this result in Section 3 in order to solve:

**Question B.** If $R$ is a locally factorial Krull domain with field of fractions $K$, give necessary and sufficient conditions on $R$ such that all maximal $R$-orders in $M_n(K)$ are conjugated.

## 1. Preliminaries

Throughout, we will consider the following situation. $R$ is a Krull domain with field of fractions $K$ and $\Lambda$ is a maximal $R$-order in some central simple algebra $\Sigma$ over $K$.

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With $O_R$ (resp. $O_\Lambda$). We will denote the structure sheaf of $R$ (resp. $\Lambda$) over Spec($R$). Our first objective is the introduction of the sheaf of normalizing elements of $\Lambda$, $N_\Lambda$. It is defined by assigning to an open set $U$ of the Zariski topology on Spec($R$) the sections

$$\Gamma(U, N_\Lambda) = N(\Gamma(U, O_\Lambda)) = \{x \in \Sigma^* : x\Gamma(U, O_\Lambda) = \Gamma(U, O_\Lambda)x\}.$$

**Proposition 1.1.** *$N_\Lambda$ is a sheaf of groups and the stalk in a prime $p$ of* Spec($R$) *equals $N(R_p)$.*

**Proof.** Let us first check that $N_\Lambda$ with inclusions as restriction morphisms is a presheaf. A typical open set of Spec($R$) is of the form $X(I) = \{p \in \text{Spec}(R): I \not\subset p\}$ for some ideal $I$ of $R$ and it is well known that $\Gamma(X(I), O_\Lambda) = Q_I(\Lambda) = \{x \in \Sigma: L \in \mathcal{L}(I) : Lx \subset \Lambda\}$ where $\mathcal{L}(I) = \{L \lhd R : I \subset \text{rad}(L)\}$. So, if $X(J) \subset X(I)$, then $\mathcal{L}(I) \subset \mathcal{L}(J)$ and we have to prove that $N(Q_I(\Lambda)) \subset N(Q_J(\Lambda))$. It follows from some results of Chamarie [1] that each $Q_I(\Lambda)$ is again a maximal order over its center which is a Krull domain and that the localization map $Q_I(\cdot)$ defines a group-epimorphism from Div($\Lambda$) onto Div($Q_I(\Lambda)$), where Div($\cdot$) is the group of divisorial ideals, cf. e.g. [1].

Thus, if $x \in N(Q_I(\Lambda))$, then there exists a divisorial $\Lambda$-ideal $A$ such that $Q_I(A) = Q_I(\Lambda)x$. Therefore, it will be sufficient to prove that $Q_J(A) = Q_J(\Lambda)x$. So, let $y \in Q_J(A)$, then there exists an ideal $K \in \mathcal{L}(J)$ such that $Ky \subset A \subset Q_I(A) = Q_I(\Lambda)x$, whence $Kyx^{-1} \subset Q_I(\Lambda) \subset Q_J(\Lambda)$ and thus $yx^{-1} \in Q_J(\Lambda)$ because every symmetric localization of $\Lambda$ is idempotent, so $y \in Q_J(\Lambda)x$. Conversely, if $y \in Q_J(\Lambda)$ then $Ky \subset \Lambda$ for some $K \in \mathcal{L}(J)$, whence $Kyx \subset \Lambda x \subset Q_I(A)$. Thus, for every $k \in K$, we can find an ideal $L \in \mathcal{L}(I) \subset \mathcal{L}(J)$ such that $Lkyx \subset A$ whence $kyx \in Q_J(A)$ and thus $Kyx \subset Q_J(A)$, yielding that $yx \in Q_J(A)$. Thus, $Q_J(A) = Q_J(\Lambda)x$ finishing the proof that $N_\Lambda$ is a presheaf, which is clearly separated. Therefore we are left to prove the gluing property. So, let $\{U_i : i \in I\}$ be an open covering of $U$ and let $x \in \Gamma(U_i, N_\Lambda)$ for every $i \in I$. Then,

$$x\Gamma(U, O_\Lambda) = x(\bigcap \Gamma(U_i, O_\Lambda)) = \bigcap \Gamma(U_i, O_\Lambda)x = \Gamma(U, O_\Lambda)x$$

whence $x \in \Gamma(U, N_\Lambda)$.

Finally, let us calculate the stalks of $N_\Lambda$ at the point $p \in \text{Spec}(R)$. Clearly, $(N_\Lambda)_p \subset N(R_p)$. Conversely, if $x \in N(R_p)$, then there exists a divisorial $\Lambda$-ideal $A$ such that $A_p = \Lambda_p x$. Thus, $(O_A)_p = \Lambda_p x$ and likewise $(O_{A^{-1}})_p = \Lambda_p x^{-1}$, where $O_A$ (resp. $O_{A^{-1}}$) is the structure sheaf of $A$ (resp. $A^{-1}$). Now, we can choose a neighborhood $V$ of $p$ such that $x \in \Gamma(V, O_A)$ and $x^{-1} \in \Gamma(V, O_{A^{-1}})$. Then,

$$x^{-1}\Gamma(V, O_\Lambda)x \subset x^{-1}\Gamma(V, O_A) \subset \Gamma(V, O_\Lambda)$$

whence $\Gamma(V, O_\Lambda)x \subset x\Gamma(V, O_\Lambda)$ and likewise one can prove the other inclusion yielding that $x \in \Gamma(V, N_\Lambda)$, finishing the proof.

The sheaf $N_\Lambda$ is not necessarily a constant sheaf, as the following example shows:

**Example 1.2..** Let $\Lambda = \mathbb{C}[X, -]$ where - denotes the complex conjugation, then $\Lambda$ is a maximal order with center $\mathbb{R}[X^2]$. In [6] it is proved that $\{X^2+c; c>0\}$ is precisely the set of the prime ideals of $\mathbb{R}[X^2]$ whose valuation extends to a valuation in $\mathbb{C}(X, -)$. If $N_\Lambda$ were constant, $N(R) = \mathbb{C}(X, -)$ yielding that every localization of $\Lambda$ at a prime ideal is a valuationring, a contradiction.

## 2. The main theorem

In this section we aim to solve question $A$, i.e. we will show how one can construct all maximal $R$-orders in a central simple algebra $\Sigma$ over $K$ from a given maximal order $\Lambda$. From [1] we retain that all maximal $R$-orders are equivalent. Of course, being conjugated defines an equivalence relation on the set of all maximal $R$-orders, so our study splits up in two cases:

I: The study of those maximal orders which are conjugated to $\Lambda$. They are of course classified by the set $\Sigma^*/N(\Lambda)$.

II: A description of the equivalence classes of nonconjugate maximal orders.

The next theorem provides such a description by means of cohomology pointed sets, cf. e.g. [2, 5].

**Theorem 2.1.** *There is a one-to-one correspondence between:*

(a) *equivalence classes of nonconjugate maximal orders,*

(b) *elements of the pointed set* $\varinjlim H^1_{\mathrm{Zar}}(U, N_\Lambda)$, *where the direct limit is taken over all open sets U of* Spec($R$) *containing* $X^1(R)$, *the set of all height one prime ideals of R.*

**Proof.** Let $\Lambda'$ be any maximal $R$-order in $\Sigma$. By $\boldsymbol{O}$ (resp. $\boldsymbol{O}'$) we denote the structure sheaf of $\Lambda$ (resp. $\Lambda'$) over Spec($R$). $\boldsymbol{T}$ (the conductor) is defined by assigning to an open set $U$ of Spec($R$) the sections

$$\Gamma(U, \boldsymbol{T}) = \{x \in \Sigma : \Gamma(U, \Lambda')x \subset \Gamma(U, \Lambda)\}.$$

First, we check that $\boldsymbol{T}$ is a sheaf. We claim that inclusions are well defined restriction morphisms. For, let $X(J) \subset X(I)$ be open sets of the Zariski topology of Spec($R$) and let $y \in \Gamma(X(I), \boldsymbol{T})$, $x \in \Gamma(X(J), \boldsymbol{O}')$, then $Lx \subset \Lambda'$ for some $L \in \mathscr{L}(J)$ whence $Lxy \subset \Gamma(X(I), \boldsymbol{O}) \subset \Gamma(X(J), \boldsymbol{O})$ entailing that $xy \in \Gamma(X(J), \boldsymbol{O})$ so $y \in \Gamma(X(J), \boldsymbol{T})$ finishing the proof of our claim. So, $\boldsymbol{T}$ is a presheaf.

Furthermore, if $U_i$ is an open covering of $U$ and if $y \in \bigcap \Gamma(U_i, \boldsymbol{T})$, then $\Gamma(U, \boldsymbol{O}')y = \bigcap \Gamma(U_i, \boldsymbol{O}')y \subset \bigcap \Gamma(U_i, \boldsymbol{O}) = \Gamma(U, \boldsymbol{O})$ proving that $y \in \Gamma(U, \boldsymbol{T})$ and therefore $\boldsymbol{T}$ is a sheaf.

For every open set U of Spec($R$), $\Gamma(U, \boldsymbol{O})$ and $\Gamma(U, \boldsymbol{O}')$ are both maximal

$\Gamma(U, \mathcal{O}_R)$-orders, hence they are equivalent. By a local application of Lemma VII.1.3 of [4] it follows that $\mathcal{T}$ is a c-$\mathcal{O}'$-$\mathcal{O}$-ideal contained both in $\mathcal{O}$ and in $\mathcal{O}'$. By this we mean that for every open set $U$, $\Gamma(U, \mathcal{T})$ is a left fractional $\Gamma(U, \mathcal{O}')$-ideal and a right fractional $\Gamma(U, \mathcal{O})$-ideal such that $(\Gamma(U, \mathcal{T})^{-1})^{-1} = \Gamma(U, \mathcal{T})$, where

$$\Gamma(U, \mathcal{T})^{-1} = \{x \in \Sigma : \Gamma(U, \mathcal{T})x \subset \Gamma(U, \mathcal{O})\} = \{x \in \Sigma : x\Gamma(U, \mathcal{T}) \subset \Gamma(U, \mathcal{O}')\}.$$

It is readily verified that $\mathcal{T}^{-1}$ which is defined by taking for its sections $\Gamma(U, \mathcal{T}^{-1}) = \Gamma(U, \mathcal{T})^{-1}$ is also a sheaf and a c-$\mathcal{O}$-$\mathcal{O}'$-ideal.

Now, let $p$ be any height one prime ideal of $R$. It is well known that $\Lambda_p$ and $\Lambda'_p$ are both principal left and right ideal rings. Therefore, there exists an invertible element $s_p$ of $\Sigma$ such that $(\mathcal{T})_p = s_p \Lambda_p$. Furthermore, $(\mathcal{T}^{-1})_p(\mathcal{T})_p = \Lambda_p$ entailing that $\Lambda_p s_p^{-1} \Lambda'_p s_p \Lambda_p = \Lambda_p$ whence $s_p^{-1} \Lambda'_p s_p \subset \Lambda_p$. By maximality of $s_p^{-1} \Lambda s_p$ this entails that $s_p^{-1} \Lambda'_p s_p = \Lambda_p$. We claim that there is a neighborhood $V(p)$ of $p$ such that $s_p^{-1}(\mathcal{O}' \mid V(p)) s_p = \mathcal{O} \mid V(p)$.

Since both $\mathcal{T}$ and $\mathcal{T}^{-1}$ are sheaves, $s_p$ and $s_p^{-1}$ live on a neighborhood $V(p)$ of $p$. Therefore, $s_p \Gamma(V(p), \mathcal{O}) \subset \Gamma(V(p), \mathcal{T})$ and $\Gamma(V(p), \mathcal{O}) s_p^{-1} \subset \Gamma(V(p), \mathcal{T}^{-1})$. Hence,

$$\begin{aligned}\Gamma(V(p), \mathcal{O}) s_p^{-1} &\subset \Gamma(V(p), \mathcal{T}^{-1}) = \Gamma(V(p), \mathcal{T})^{-1} \\ &\subset (s_p \Gamma(V(p), \mathcal{O}))^{-1} = \Gamma(V(p), \mathcal{O}) s_p^{-1}\end{aligned}$$

and therefore $\Gamma(V(p), \mathcal{T}^{-1}) = \Gamma(V(p), \mathcal{O}) s_p^{-1}$ and likewise, $\Gamma(V(p), \mathcal{T}) = s_p \Gamma(V(p), \mathcal{O})$. This then entails that $s_p^{-1}(\mathcal{O}' \mid V(p)) s_p = \mathcal{O} \mid V(p)$.

Thus, $\bigcup V(p)$ is an open set containing $X^1(R)$. Now, $X^1(R)$ equipped with the induced Zariski topology is a Noetherian space and therefore we can find a finite number among these $V(p)$, say $V(p_1), \dots, v(p_n)$ such that $U = \bigcup V(p_i)$ contains $X^1(R)$.

For any $i, j \in 1, \dots, n$ we have that

$$s_{p_i}(\mathcal{O} \mid V(p_i) \cap V(p_j)) s_{p_i}^{-1} = s_{p_j}(\mathcal{O} \mid V(p_i) \cap V(p_j)) s_{p_j}^{-1}$$

and this entails that $s_{p_i}^{-1} s_{p_j} \in \Gamma(V(p_i) \cap V(p_j), N_\Lambda)$. Therefore $\{V(p_i), s_{p_i}\}$ describes a section of $\Gamma(U, \Sigma^*/N_\Lambda)$. Now consider the exact sequence of sheaves of pointed sets

$$1 \to N_\Lambda \to \Sigma^* \to \Sigma^*/N_\Lambda \to 1.$$

Taking sections over $U$ yields the exact sequence of pointed sets

$$1 \to N(\Lambda) \to \Sigma^* \to \Gamma(U, \Sigma^*/N_\Lambda) \to H^1_{\mathrm{Zar}}(U, N_\Lambda) \to 1.$$

Therefore, the section $\{V(p_i), s_{p_i}\}$ determines an element in $H^1_{\mathrm{Zar}}(U, N_\Lambda)$ (and thus also in $\lim H^1_{\mathrm{Zar}}(U, N_\Lambda)$) which differs from the distinguished element in $H^1_{\mathrm{Zar}}(U, N_\Lambda)$ if and only if $\Lambda'$ is not conjugated to $\Lambda$.

Conversely, let $s \in \lim H^1_{\mathrm{Zar}}(U, N_\Lambda)$ and choose an open set $U$ of Spec($R$) containing $X^1(R)$ and an element $s(U) \in H^1_{\mathrm{Zar}}(U, N_\Lambda)$ which represents $s$. Using the above exact sequence, $s(U)$ is determined by some section in $\Gamma(U, \Sigma^*/N_\Lambda)$. Such a section is given by a set of couples $\{(U_i, s_i)\}$ where $U_i$ is an open covering of $U$,

$s_i \in \Gamma(U_i, \Sigma^*)$ for every $i$ and for all $i$ and $j$ and we have that $s_i^{-1}s_j \in \Gamma(U_i \cap U_j, N_\Lambda)$. On $U$ we will define the twisted sheaf of maximal orders $\boldsymbol{O}' \mid U$ by putting $\boldsymbol{O}' \mid U_i = s_i(\boldsymbol{O} \mid U_i)s_i^{-1}$. Using the fact that $s_i^{-1}s_j \in \Gamma(U_i \cap U_j, N_\Lambda)$ it is easily verified that this is indeed a sheaf. We claim that $\Lambda' = \Gamma(U, \boldsymbol{O}' \mid U)$ is a maximal $R$-order.

Firstly we will show that there exists an open refinement $\{W_k\}$ of $\{U_i\}$ and sections $t_k \in \Gamma(W_k, \Sigma^*)$ such that $t_k^{-1}t_1 \in \Gamma(W_k \cap W_1, \boldsymbol{O}^*)$ and with the property that the twisted sheaf of maximal orders determined by $(W_k, t_k)$ coincides with $\boldsymbol{O}'$ on $\bigcup W_k$. Because $X^1(R)$ is a Noetherian space, there are a finite number among the $U_i$, say $U_1, \dots, U_n$ such that $U' = \bigcup U_i \supset X^1(R)$. For any $i, j$ among $1, \dots, n$, $Z(i,j) = \{p \in U_i \cap U_j : s_i^{-1}s_j \notin \Lambda_p\}$ is a finite set, because $\mathrm{Div}(\Gamma(U, \boldsymbol{O}))$ is the free abelian group generated by $X^1(R) \cap U$ for any open set $U$. So, $Z(1) = Z(1,2) \cup Z(1,3) \cup \cdots \cup Z(1,n)$ is a finite set. Now because the Zariski topology induced on $X^1(R)$ is the cofinite topology, there exists an open $V$ in $\mathrm{Spec}(R)$ such that $V \cap X^1(R) = X^1(R)/Z(1)$. Take $W_1 = U_1 \cap V$, $W_i = U_i$, for $i \neq 1$, $t_1 = s_1 \mid W_1$ and $t_i = s_i$ for $i \neq 1$, then $t_1^{-1} \cdot t_j \in \Gamma(W_1 \cap W_j, \boldsymbol{O}^*)$. Continuing in this manner we will eventually find $(W_k, t_k)$ satisfying the requirements, in particular, if $W = \bigcup W_k$, then $\boldsymbol{O}' \mid W$ coincides with the twisted sheaf of maximal orders determined by the $t_k$.

Next we define a sheaf $\boldsymbol{T} \mid W$ by $\boldsymbol{T} \mid W_k = t_k(\boldsymbol{T} \mid W_k)$. Clearly, $\boldsymbol{T} \mid W$ is a right $\boldsymbol{O}$-ideal and $(\boldsymbol{T} \mid W)^{-1})^{-1} = \boldsymbol{T} \mid W$, this yields that for every open $V \subset W$, $\Gamma(V, \boldsymbol{O})$ is a right fractional c-$\Gamma(V, \boldsymbol{O})$-ideal. This implies that $O_1(\Gamma(V, \boldsymbol{T})) = \Gamma(V, \boldsymbol{O}' \mid W)$ is a maximal order.

In particular, $\Gamma(W, \boldsymbol{O}' \mid W) = \Gamma(U, \boldsymbol{O}' \mid U)$ is a maximal order.

Finally, the reader may check that the constructions above do not depend on the choices made.

**Corollary 2.2.** *If $R$ is a Dedekind domain, there is a one-to-one correspondence between:*

(a) *equivalence classes of non-conjugate maximal orders,*

(b) *elements of $H^1_{\mathrm{Zar}}(X, N_\Lambda)$.*

## 3. Application: maximal orders in matrixrings

In this section we aim to characterize those locally factorial (i.e. $R_p$ is a UFD for every $p \in \mathrm{Spec}(R)$) Krull domains for which all maximal orders in $M_n(K)$ are conjugated. In this situation we are able to compute $H^1_{\mathrm{Zar}}(U, N_\Lambda)$ for $\Lambda = M_n(R)$.

With $\mathbf{PGL}_n$ we will denote $\mathbf{Aut}(P^n_R)$, the automorphism scheme of the $n$-dimensional projective space over $R$, i.e. $\mathbf{PGL}_n$ is the sheafification of the presheaf which assigns $\mathrm{PGL}_n(\Gamma(U, \boldsymbol{O}_R))$ to any open set of $\mathrm{Spec}(R)$, cf. e.g. [5].

**Proposition 3.1.** *If $R$ is a locally factorial Krull domain and if $\Lambda = M_n(R)$, then $H^1_{\mathrm{Zar}}(U, N_\Lambda) = H^1_{\mathrm{Zar}}(U, \mathbf{PGL}_n)$ for every open set $U$ of* $\mathrm{Spec}(R)$.

**Proof.** If we assign to an open set $U$ of Spec($R$) the group $\mathrm{GL}_n(\Gamma(U, O_R)) \cdot K^* \subset \mathrm{GL}_n(K)$, then this defines a presheaf of groups. Its sheafification will be denoted by $\mathbf{GL}_n \cdot \boldsymbol{K}^*$. This sheaf is clearly a subsheaf of $N_\Lambda$. We will show that their stalks are isomorphic. If $p \in \mathrm{Spec}(R)$ and if $x \in N(M_n(R_p))$, then $M_n(R)x = M_n(A)$ for some divisorial $R_p$-ideal $A$. Because $R_p$ is a UFD, $A = R_p \cdot k$ for some $k \in K^*$, yielding that $x \in \mathrm{GL}_n(R_p) \cdot K^*$ proving that $\mathbf{GL}_n \cdot \boldsymbol{K}^* = N_\Lambda$.

The following sequence of sheaves of groups is exact:

$$1 \to \boldsymbol{K}^* \to \mathbf{GL}_n \cdot \boldsymbol{K}^* \to \mathbf{PGL}_n \to 1$$

where $\boldsymbol{K}^*$ denotes the constant sheaf associated with $K^*$.

Taking sections over $U$ yields the following long exact cohomology sequence:

$$1 \to \Gamma(U, \boldsymbol{K}^*) \to \Gamma(U, N_\Lambda) \to \Gamma(U, \mathbf{PGL}_n)$$
$$\to 1 \to H^1_{\mathrm{Zar}}(U, N_\Lambda) \to H^1_{\mathrm{Zar}}(U, \mathbf{PGL}_n) \to 1,$$

finishing the proof.

*A. Dedekind domains*

**Proposition 3.2.** *If $R$ is a Dedekind domain, then all maximal $R$-orders in $M_n(K)$ are conjugated if and only if $(-)^n : \mathrm{Cl}(R) \to \mathrm{Cl}(R)$ sending $[A]$ to $[A^n]$ is an epimorphism.*

**Proof.** In view of Corollary 2.2 and Proposition 3.1 we have to find an equivalent condition for $H^1_{\mathrm{Zar}}(X, \mathbf{PGL}_n) = 1$. Writing out the long exact cohomology sequence of the following exact sequence of sheaves of groups

$$1 \to O_R^* \to \mathbf{GL}_n \to \mathbf{PGL}_n \to 1$$

entails

$$H^1_{\mathrm{Zar}}(X, O_R^*) \xrightarrow{\delta} H^1_{\mathrm{Zar}}(X, \mathbf{GL}_n) \to H^1_{\mathrm{Zar}}(X, \mathbf{PGL}_n) \to H^2_{\mathrm{Zar}}(X, O_R^*).$$

Because $R$ is a Dedekind domain (Krull dimension $= 1$) $H^2_{\mathrm{Zar}}(X, O_R^*) = 1$. Furthermore, $H^1_{\mathrm{Zar}}(X, \mathbf{GL}_n)$ is the set of isomorphism classes of projective rank $n$ $R$-modules, which we denote by $\mathrm{Proj}_n(R)$. By Steinitz' result any projective rank $n$ module is isomorphic to $J_1 \oplus \cdots \oplus J_n$ for some fractional $R$-ideals $J_i$ and $\delta$ is epimorphic if and only if there exists a fractional $R$-ideal $I$ such that $J_1 \oplus \cdots \oplus J_n \cong I \oplus \cdots \oplus I$ yielding that $J_1 \cdots J_n \cong I^n$, finishing the proof.

**Remark 3.3.** F. Van Oystaeyen suggested a more ringtheoretical proof of this result in the following way. Because all maximal $R$-orders in $M_n(K)$ are Morita equivalent and $M_n(R)$ is Azumaya, they are all Azumaya algebras. Furthermore $\mathrm{Br}(R) \subset \mathrm{Br}(K)$ whence any maximal order is of the form $\mathrm{End}_R(P)$ where $P \in \mathrm{Proj}_n(R)$. Applying again Steinitz' theorem to the condition $\mathrm{End}_R(P) \cong M_n(R)$ yields the same condition on $\mathrm{Cl}(R)$.

*B. Regular local domains*

We recover the classical result of M. Ramas for matrixrings:

**Proposition 3.4.** *If $R$ is a regular local ring of* gldim$(R)\leq 2$, *then all maximal orders in $M_n(K)$ are conjugated.*

**Proof.** We have to check that $H^1_{\mathrm{Zar}}(U,\mathbf{PGL}_n)=1$ where $U=X(m)$, $m$ being the maximal ideal of $R$. Again consider the exact sequence

$$H^1_{\mathrm{Zar}}(U,O_R^*)\to H^1_{\mathrm{Zar}}(U,\mathbf{GL}_n)\to H^1_{\mathrm{Zar}}(U,\mathbf{PGL}_n)\to H^2_{\mathrm{Zar}}(U,O_R^*).$$

Now, $H^1_{\mathrm{Zar}}(U,\mathbf{GL}_n)$ is the set of isomorphism classes of reflexive $R$-modules which are free of rank $n$ at every height one prime ideal of $R$, $\mathrm{Ref}_n(R)$. Because $\mathrm{gldim}(R)\leq 2$, reflexive modules are projective whence $\mathrm{Ref}_n(R)=\mathrm{Proj}_n(R)$ and $\mathrm{Ref}_1(\mathrm{R})=\mathrm{Pic}(R)$. Finally, $R$ being local $\mathrm{Pic}(R)=\mathrm{Proj}_n(R)=1$ and therefore all cohomology pointed sets above are trivial except perhaps $H^1_{\mathrm{Zar}}(U,\mathbf{PGL}_n)$ but exactness of the sequence finishes the proof.

*C. Locally factorial Krull domains*

**Theorem 3.5.** *If $R$ is a locally factorial Krull domain then all maximal orders in $M_n(K)$ are conjugated if and only if the map from* $\mathrm{Cl}(R)$ *to* $\mathrm{Ref}_n(R)$ *sending* $[I]$ *to* $[I\oplus\cdots\oplus I]$ *is surjective.*

**Proof.** Consider the exact sequence

$$\lim H^1(U,O_R^*)\to\lim H^1(U,\mathbf{GL}_n)\to\lim H^1(U,\mathbf{PGL}_n)\to\lim H^2(U,O_R^*)$$

where the direct limit is taken over all opens $U$ containing $X^1(R)$.

Because $R$ is locally factorial, Cartier divisors coincide with Weil divisors showing that the sequence

$$1\to O_R^*\to K\to\mathbf{Div}\to 1$$

is exact. Because the sheaf of Weil divisors, **Div**, is flabby, $H^2_{\mathrm{Zar}}(U,O_R^*)=1$ for any open set $U$ showing that the last term in the sequence vanishes.

So, by Theorem 2.1 and Proposition 3.1 all maximal orders in $M_n(K)$ are conjugated iff the map from $\lim H^1(U,O_R^*)=\mathrm{Cl}(R)$ to $\lim H^1(U,\mathbf{GL}_n)=\mathrm{Ref}_n(R)$ which is defined by sending a class of a divisorial ideal $[I]$ to $[I\oplus\cdots\oplus I]$ is surjective.

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